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J. Math. Anal. Appl. 277 (2003) 155–163
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Entire functions that share one value CM with their derivatives ☆

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Received 8 November 2001
Submitted by S.-C. Chen

Abstract

This paper studies the uniqueness problem on entire function that share a finite, nonzero value CM with their derivatives and proves two main theorems which generalize some results given by Jank, Mues and Volkmann, P. Li and C.C. Yang, H.L. Zhong etc. An example shows that the condition of one of our theorems is necessary.

Keywords: Entire function; Sharing value; Uniqueness

1. Introduction and main results

Let f and g be two nonconstant meromorphic functions in the complex plane, and let a be a finite value. We say that f and g share the value a CM provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a IM provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value distribution theory (see [4] or [11]). As usual, we denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside a set of r of finite linear measure. In addition, for a meromorphic function λ and a positive integer k, we

☆ Project supported by the NSF and the RFDP of China.
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0022-247X/02/$ – see front matter
PII: S0022-247X(02)00526-7

denote by $P_k[\lambda]$ or $P_k^*[\lambda]$ or $Q_k[\lambda]$ a polynomial in λ and its derivatives with constant coefficients and degree at most k, not necessarily the same at each occurrence.

Rubel and Yang proved the following theorem.

Theorem A [8]. *Let f be a nonconstant entire function. If f and f' share two finite, distinct values CM, then $f \equiv f'$.*

In 1986, Jank, Mues and Volkmann proved the next result.

Theorem B [6]. *Let f be a nonconstant entire function. If f and f' share a finite, nonzero value a IM, and if $f''(z) = a$ whenever $f(z) = a$, then $f \equiv f'$.*

Remark 1. From the hypothesis of Theorem B, it can be easily seen that the value a is shared by f and f' CM.

The following counterexample (see [13]) shows that Theorem B is, in general, not true if the f'' of Theorem B is replaced by $f^{(k)}$ ($k \geqslant 3$).

Let k ($\geqslant 3$) be a positive integer, and let a be a $(k-1)$th root of unity satisfying $a \neq 1$. Set $f(z) = e^{az} + a - 1$. It is easy to know that f, f' and $f^{(k)}$ share the value a CM, but $f \not\equiv f'$ and $f \not\equiv f^{(k)}$.

In 1995, H.L. Zhong proved the following result.

Theorem C [13]. *Let f be a nonconstant entire function, and let n be a positive integer. If f and f' share a finite, nonzero value a CM, and if $f^{(n)}(z) = f^{(n+1)}(z) = a$ whenever $f(z) = a$, then $f \equiv f^{(n)}$.*

As we known, one of the shared problems has been the case of f and f' sharing values, and some interesting results on this topic have been obtained (see, for example, [2,6,7,9, 10,12,13], etc). A natural question is:

Question 1. What can be said when the f'' of Theorem B is replaced by $f^{(k)}$ ($k \geqslant 3$)?

The purpose of this paper is to solve the above question by giving the definite expression of f, and proving that the above counterexample is unique. Moreover, we also generalize Theorem C and improve Theorem D which will be stated later.

Theorem 1. *Let f be a nonconstant entire function, let a ($\neq 0$) be a finite constant, and let n and m be positive integers satisfying $m > n$. If f and f' share the value a CM, and if $f^{(m)}(z) = f^{(n)}(z) = a$ whenever $f(z) = a$, then*

$$f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda},$$

where A ($\neq 0$) and λ are constants satisfying $\lambda^{n-1} = 1$ and $\lambda^{m-1} = 1$.

Remark 2. Under the hypothesis of Theorem 1, we must have $f' \equiv f^{(n)} \equiv f^{(m)}$. In Theorem 1, if $n = 1$ and $m = 2$, then we have $\lambda = 1$ which implies $f \equiv f'$. So Theorem 1 contains Theorem B. In Theorem 1, if $m = n + 1$, then we have $\lambda = 1$ which implies $f \equiv f^{(n)}$. So Theorem 1 contains also Theorem C.

From Theorem 1, we can obtain the following corollary immediately.

Corollary 1. *Let f be a nonconstant entire function, let a $(\neq 0)$ be a finite constant, and let m and n be positive integers satisfying $m > n = 2$ or $m - 1$ and $n - 1$ are relatively prime when $m > n \geqslant 3$. If f and f' share the value a CM, and if $f^{(m)}(z) = f^{(n)}(z) = a$ whenever $f(z) = a$, then $f \equiv f' \equiv f^{(n)} \equiv f^{(m)}$.*

Remark 3. In Theorem 1, set $f(z) = e^{-z} + 2$, take $m = 5$, $n = 3$, $a = 1$. It is easy to see that all the conditions of Theorem 1 are satisfied by $f(z)$, but $f \not\equiv f'$. This example shows that the hypothesis "two positive integers $m - 1$ and $n - 1$ are relatively prime" in Corollary 1 is necessary to the conclusion $f \equiv f'$ in the case $m > n \geqslant 3$.

Theorem B suggests the following Question of Yi and Yang.

Question 2 (see [11, p. 458] or [5]). Let f be a nonconstant meromorphic function, let a be a finite, nonzero constant, and let n and m $(n < m)$ be positive integers. If f, $f^{(n)}$ and $f^{(m)}$ share a CM, where n and m are not both even or both odd, must $f \equiv f^{(n)}$?

An example (see [9]) given by Yang shows that the answer to the above Question 2 is, in general, negative. Very recently, related to Question 2, Li and Yang obtained the following theorem.

Theorem D [7]. *Let f be an entire function, let a be a finite nonzero value, and let n $(\geqslant 2)$ be a positive integer. If f, f', and $f^{(n)}$ share the value a CM, then f assumes the form*

$$f(z) = be^{cz} + a - \frac{a}{c},$$

where b, c are nonzero constants and $c^{n-1} = 1$.

We prove the following result.

Theorem 2. *Let f be a nonconstant entire function, let a $(\neq 0)$ be a finite constant, and let k $(\geqslant 2)$ be an integer. If f and f' share a CM, and if $f^{(k)}(z) = a$ whenever $f(z) = a$, then f assumes the form*

$$f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda},$$

where A $(\neq 0)$ and λ are constants satisfying $\lambda^{k-1} = 1$.

Remark 4. Under the hypothesis of Theorem 2, we must have $f' \equiv f^{(k)}$. In Theorem 2, if $k = 2$, then we have $\lambda = 1$ which implies $f \equiv f'$. So Theorem 2 contains Theorem B. It is easy to see that Theorem 2 is a special case of Theorem 1 with $n = 1$ and $m = k$. Obviously, Theorem 2 answers Question 1, and Theorem 2 has also improved Theorem D.

2. Lemmas

Lemma 1. *Let f be a transcendental meromorphic solution of the equation*

$$f^n P(f) = Q(f),$$

where $P(f)$ and $Q(f)$ are polynomials in f and its derivatives with meromorphic coefficients, say a_j. If the total degree of Q is at most n, then

$$m\bigl(r, P(f)\bigr) \leqslant \sum_j m(r, a_j) + S(r, f).$$

Lemma 1 is essentially due to Clunie [3]. In fact, the proof of Lemma 1 is a simple modification of the proof of Lemma 3.3 in [4], see also [1, Lemma 1].

Lemma 2 [10, Theorem 2]. *Let f be a nonconstant entire function of finite order, let a $(\neq 0)$ be a finite value, and let k be a positive integer. If f and $f^{(k)}$ share the value a CM, then*

$$\frac{f^{(k)} - a}{f - a} \equiv c,$$

for some nonzero constant c.

To prove our theorems, we need the following result, which is interesting by itself.

Lemma 3. *Let f be a nonconstant entire function, let a $(\neq 0)$ be a finite constant, and let m $(\geqslant 2)$ be a positive integers. If $m(r, 1/(f - a)) = S(r, f)$, and if $f'(z) = f^{(m)}(z) = a$ whenever $f(z) = a$, then*

(i) $f(z) = Ae^{\lambda z} + a - a/\lambda$, *where A $(\neq 0)$ and λ are constants satisfying $\lambda^{m-1} = 1$;*
(ii) $f' \equiv f^{(m)}$.

Proof. Set

$$\lambda = \frac{f' - a}{f - a}. \tag{2.1}$$

From the hypothesis of Lemma 3, we know that λ is an entire function satisfying $T(r, \lambda) = S(r, f)$. Rewrite (2.1) as

$$f' = \lambda f + a(1 - \lambda) := \lambda_1 f + \mu_1, \tag{2.2}$$

where λ_1 and μ_1 are defined by

$$\lambda_1 = \lambda, \qquad \mu_1 = a(1 - \lambda). \tag{2.3}$$

By taking the derivatives on both sides of (2.2), we get

$$f^{(k)} = \lambda_k f + \mu_k \tag{2.4}$$

for $k = 1, 2, \ldots$, where λ_k and μ_k are entire functions satisfying the following recurrence formulas

$$\lambda_{k+1} = \lambda_k' + \lambda_1\lambda_k \quad \text{for } k = 1, 2, \ldots \tag{2.5}$$

and

$$\mu_{k+1} = \mu_k' + \mu_1\lambda_k \quad \text{for } k = 1, 2, \ldots. \tag{2.6}$$

From (2.1), (2.3), (2.5) and (2.6), we have

$$T(r, \lambda_k) = S(r, f), \qquad T(r, \mu_k) = S(r, f) \quad \text{for } k = 1, 2, \ldots.$$

Noting that $m(r, 1/(f-a)) = S(r, f)$, thus $N(r, 1/(f-a)) \neq S(r, f)$. Now, we can assume that if $f(z_0) = a$, then we have $f^{(m)}(z_0) = a$ from the hypothesis of Lemma 3. It follows from (2.4) that

$$a = a\lambda_m(z_0) + \mu_m(z_0).$$

Since $N(r, 1/(f-a)) \neq S(r, f)$, thus we must have

$$a \equiv a\lambda_m + \mu_m. \tag{2.7}$$

From (2.3), (2.5) and (2.6), and by induction in the number k, it can be easily obtained that

$$\lambda_k = \lambda^k + P_{k-1}[\lambda] \quad \text{for } k = 1, 2, \ldots \tag{2.8}$$

and

$$\mu_k = -a\lambda^k + P^*_{k-1}[\lambda] \quad \text{for } k = 1, 2, \ldots. \tag{2.9}$$

Again applying induction in the number k, we get

$$\mu_{k+1} + a\lambda_{k+1} = a\lambda^k + Q_{k-1}[\lambda] \quad \text{for } k = 1, 2, \ldots. \tag{2.10}$$

Observing that differentiation never increases the degree of a differential polynomial, we may prove (2.10) by induction as follows. By (2.3), (2.5) and (2.6), a simple calculation gives $\mu_2 + a\lambda_2 = a\lambda$. Suppose now that

$$\mu_k + a\lambda_k = a\lambda^{k-1} + Q_{k-2}[\lambda] \tag{2.11}$$

has been proved. By (2.3), (2.5), (2.6), (2.8) and (2.9), we now obtain

$$\begin{aligned}\mu_{k+1} + a\lambda_{k+1} &= \left(-a\lambda^k + P^*_{k-1}[\lambda]\right)' + a(1-\lambda)\lambda_k + a\left(\left(\lambda^k + P_{k-1}[\lambda]\right)' + \lambda\lambda_k\right)\\ &= -ka\lambda^{k-1}\lambda' + \left(P^*_{k-1}[\lambda]\right)' + a\lambda_k - a\lambda\lambda_k\\ &\quad + ka\lambda^{k-1}\lambda' + a\left(P_{k-1}[\lambda]\right)' + a\lambda\lambda_k\\ &= a\lambda^k + aP_{k-1}[\lambda] + a\left(P_{k-1}[\lambda]\right)' + \left(P^*_{k-1}[\lambda]\right)'\\ &= a\lambda^k + Q_{k-1}[\lambda],\end{aligned}$$

which proves (2.10).

From (2.7) and (2.10), we obtain

$$a\lambda^{m-1} + Q_{m-2}[\lambda] \equiv a. \tag{2.12}$$

Obviously, $Q_{m-2}[\lambda] \not\equiv a$, otherwise, we will get a contradiction from (2.12), (2.1) and the hypothesis of Lemma 3. By (2.12) and Lemma 1, we can deduce that λ is a constant. Noting that f' is nonconstant, so $\lambda \neq 0$. Thus by (2.5), we have

$$\lambda_k = \lambda^k \quad \text{for } k = 1, 2, \ldots. \tag{2.13}$$

Since $\mu_1 = a - a\lambda$, it follows from (2.13) and (2.6) that

$$\mu_k = a(1-\lambda)\lambda^{k-1} \quad \text{for } k = 1, 2, \ldots. \tag{2.14}$$

From (2.7), (2.13) and (2.14), we have

$$a = a\lambda^m + a(1-\lambda)\lambda^{m-1},$$

which gives

$$\lambda^{m-1} = 1. \tag{2.15}$$

Furthermore, by (2.1), we can obtain

$$f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}, \tag{2.16}$$

where A is an arbitrary nonzero constant. From (2.15) and (2.16) we can immediately deduce that $f' \equiv f^{(m)}$. The proof of Lemma 3 is complete. □

3. The proof of Theorem 1

Set

$$\alpha = \frac{f^{(m)} - f'}{f - a}, \tag{3.1}$$

$$\beta = \frac{f^{(n)} - f'}{f - a}. \tag{3.2}$$

We know from the hypothesis of Theorem 1 that α and β are all entire functions satisfying $T(r,\alpha) = S(r, f)$ and $T(r,\beta) = S(r, f)$. We shall divide our argument into three cases.

Case 1. Suppose that $\alpha \not\equiv 0$. By (3.1) we have

$$f = a + \frac{1}{\alpha}\bigl(f^{(m)} - f'\bigr). \tag{3.3}$$

Taking the derivatives on both sides of (3.3) gives

$$\left[1 + \left(\frac{1}{\alpha}\right)'\right] f' = \left(\frac{1}{\alpha}\right)' f^{(m)} + \frac{1}{\alpha}\bigl(f^{(m+1)} - f''\bigr).$$

Since α is an entire function, thus $1 + (1/\alpha)' \not\equiv 0$. From the above equality, we get

$$\frac{f'}{f' - a} = \frac{(1/\alpha)'}{1 + (1/\alpha)'} \frac{f^{(m)}}{f' - a} + \frac{\frac{1}{\alpha}}{1 + (1/\alpha)'} \frac{f^{(m+1)} - f''}{f' - a}. \tag{3.4}$$

By (3.4),

$$m\left(r, \frac{f'}{f'-a}\right) = S(r, f). \tag{3.5}$$

Since $f' = (f'-a) + a$, by using (3.5) we obtain

$$m\left(r, \frac{1}{f'-a}\right) = m\left(r, \frac{a}{f'-a}\right) + O(1) \leqslant m\left(r, \frac{f'}{f'-a}\right) + O(1) = S(r, f). \tag{3.6}$$

Moreover, from (3.3) we have

$$\begin{aligned} T(r, f) = m(r, f) = m\left(r, a + \frac{f^{(m)} - f'}{\alpha}\right) \\ \leqslant m(r, f') + S(r, f) = T(r, f') + S(r, f) \leqslant T(r, f) + S(r, f). \end{aligned} \tag{3.7}$$

From (3.7), we get

$$T(r, f) = T(r, f') + S(r, f). \tag{3.8}$$

Since the value a is shared by f and f' CM, by using (3.6), (3.8) together with the first main theorem, we have

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &= T(r, f) - N\left(r, \frac{1}{f-a}\right) + O(1) \\ &= T(r, f') - N\left(r, \frac{1}{f-a}\right) + S(r, f) \\ &= N\left(r, \frac{1}{f'-a}\right) - N\left(r, \frac{1}{f-a}\right) + S(r, f) = S(r, f), \end{aligned}$$

From this and Lemma 3, we can derive $f' \equiv f^{(m)}$. Thus, $\alpha \equiv 0$, which is a contradiction.

Case 2. Suppose that $\beta \not\equiv 0$. In the same manner as Case 1, we can obtain $f' \equiv f^{(n)}$. Thus, $\beta \equiv 0$, which is also a contradiction.

Case 3. Suppose that $\alpha \equiv 0$ and $\beta \equiv 0$. It follows from (3.1) that $f' \equiv f^{(m)}$, which implies that f is an entire function of finite order. In fact, by solving the equation $f' \equiv f^{(m)}$, we find that $f(z) = C_0 + C_1 e^{t_1 z} + \cdots + C_{m-1} e^{t_{m-1} z}$, where $t_1, \ldots, t_{m-1}$ are distinct $(m-1)$th roots of unity and $C_0, C_1, \ldots, C_{m-1}$ are constants. Since f and f' share the value a CM, by Lemma 2, we have

$$\frac{f'-a}{f-a} = \lambda \tag{3.9}$$

for some nonzero constant λ. From (3.9) we get

$$f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}, \tag{3.10}$$

where A is an arbitrary nonzero constant. The fact $f' \equiv f^{(m)}$, which when combined with (3.10) gives

$$\lambda^{m-1} = 1. \tag{3.11}$$

Assume that $n = 1$. Obviously,

$$\lambda^{n-1} = 1. \tag{3.12}$$

From (3.10), (3.11) and (3.12), we can derive the conclusion of Theorem 1.

Assume that $n \geqslant 2$. From (3.10) and $\beta \equiv 0$ we have also

$$\lambda^{n-1} = 1. \tag{3.13}$$

From (3.10), (3.11) and (3.13), we can derive the conclusion of Theorem 1.

The proof of Theorem 1 is then completed.

4. The proof of Theorem 2

Set

$$H = \frac{f^{(k)} - f'}{f - a}. \tag{4.1}$$

If $H \equiv 0$, then from (4.1) we have $f' \equiv f^{(k)}$, which implies that f is an entire function of finite order. Noting that f and f' share the value a CM, it follows from Lemma 2 that

$$\frac{f' - a}{f - a} = \lambda \tag{4.2}$$

for some nonzero constant λ. By (4.2), we have

$$f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}, \tag{4.3}$$

where A ($\neq 0$) is an arbitrary constant. From (4.3) and the fact that $f' \equiv f^{(k)}$ we can deduce that $\lambda^{k-1} = 1$.

If $H \not\equiv 0$, we can deal with it as the Case 1 in the proof of Theorem 1, and then from Lemma 3 deduce a contradiction. This completes the proof of Theorem 2.

Acknowledgment

The authors would like to thank the referee for his/her thorough review and valuable suggestions towards improvement of the paper.

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